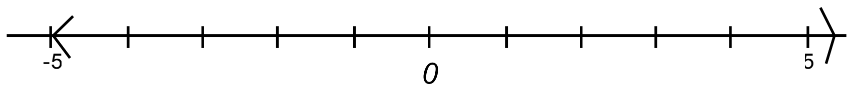
Section 3-6, Mathematics 104

The real number line and intervals

We often draw the real number line as follows:



An **interval** is a piece of this number line.

A typical interval is the [0, 5]

This interval includes all the number between 0 and 5, including 0 and 5.

This is also called a **closed interval**.

Another way to write this is:

$0 \le x \le 5$

If we want to exclude the end points 0, and 5 we can write it as follows

(0,5)

or

$0 < x < 5$

This is called an **open interval**.

An interval can also be half open and closed, for example:

(3, 7] or [5, 9)

These intervals are also called **bounded.**

An **unbounded interval** is one that continues on to infinity in one direction.
We always indicate the un-bounded direction as open.

$[0,\infty)$ or $(-\infty,-1]$

Note: the unbounded interval $(-\infty,\infty)$ is the entire real number line.

Recall that the length of any interval $[a,b] = b - a$

Inequalities

An inequality is like an expression except that the equal sign becomes an inequality sign.

The four such signs are

$<$ - Less than
$>$ - Greater than
$\leq$ - Less than or equal to
$\geq$ - Greater than or equal to

An inequality can be very simple, for example:

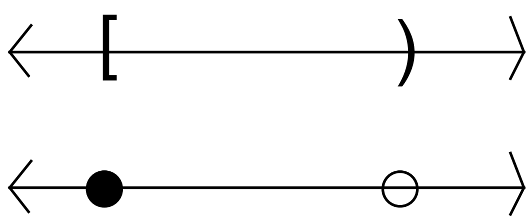
$x < 10$

As previously mentioned, we can write two inequalities together to describe an interval,

$-6 < x < 15$ means

$x > -6$ and $x < 15$

Graphing an inequality we can use the [and) symbols to indicate open and closed intervals or we can use filled in or empty circles.



Our book seems to prefer the former but either is acceptable.

Solving a linear inequality

Solving equalities, we found that we can add, subtract, multiply and divide to both sides of an equation while keeping the equality.   Things are slightly different for an inequality.

$$
\begin{aligned}
&5 < 10 \\
&5 + 20 < 10 + 20 \\
&25 < 30
\end{aligned}
$$

Note that adding or subtracting from both sides does not change the inequality.

$$
\begin{aligned}
&5 < 10 \\
&5(3) < 10(3) \\
&15 < 30
\end{aligned}
$$

Note that multiplying doesn't seem to change the inequality either, however

$$
\begin{aligned}
&5 < 10 \\
&5(-2) < 10(-2) \\
&-10 \not< -20 \\
&-10 > -20
\end{aligned}
$$

It seems that when multiplying (or dividing) by a negative number the inequality sign switches direction.

A $<$ changes to $>$
A $>$ changes to $<$
A $\le$ changes to $\ge$
A $\ge$ changes to $\le$

And of course if you multiply both sides by zero, you get an obvious equality 0=0

Example:

$x + 6 < 9$

$x < 9-6$

$x < 3$

Example:

$$8-3x \leq 20$$
$$-3x \leq 12$$
$$x \geq -4$$

Note the inequality sign switches

Example:

$$7x-3 > 3(x+1)$$
$$7x-3 > 3x+3$$
$$4x > 6$$
$$x > \frac{6}{4}$$
$$x > \frac{3}{2}$$

Example:
Note how we solve two different inequalities at the same time.

$$-7 \leq 5x-2 < 8$$
$$-5 \leq 5x < 10$$
$$-1 \leq x < 2$$

Example:
A compound inequality

$$-3x+6 \leq 2 \qquad -3x+6 \geq 7$$
$$-3x \leq -4 \qquad -3x \geq 1$$
$$x \geq \frac{4}{3} \qquad x \leq -\frac{1}{3}$$

Section 3-7, Mathematics 104

Absolute Value Equations and Inequalities

I'd like to note here, for many students, this is one of the hardest sections to get right.

We would like to add to our ability to solve equations and inequalities but include in these statements expressions with inequalities.

One feature that is important to note throughout is that when you have an inequality, you create two statements, one for when the contents of the inequality are greater than or equal to zero, and one for when it is less than zero.

Recall that

$$|x| = x \text{ for } x \geq 0$$

$$|x| = -x \text{ for } x < 0$$

Example:

$$|x| = 10$$

for $x \geq 0$ $|x| = x = 10 \rightarrow x = 10$
for $x < 0$ $|x| = -x = 10 \rightarrow x = -1$

So x=10 and x=-10 are both solutions.

Plug them both back in to check.

$$|x| = 0$$

In this case there is only one solution x=0.

$$|x| = -1$$

Think about this for a second. Of course there are no solutions.

Inequalities with two absolute values

How would we go about this?

$$|3x-4|=|7x-16|$$

The two expressions inside the absolute value can each be greater than or equal to zero or less than zero.

There are four cases here, but only two equations to solve.

$$3x-4=7x-16$$

if both expressions are greater than or equal to zero or both less than zero.

$$12=4x$$
$$x=3$$

if one expression is greater than or equal to zero and the other is less we have

$$3x-4=-(7x-16)$$
$$3x-4=-7x+16$$
$$10x=20$$
$$x=2$$

It is important to plug both these values in to check

$$|3(3)-4|=|7(3)-16|$$
$$|9-4|=|21-16|$$
$$|5|=|5|$$
$$5=5$$

$$|3(2)-4|=|7(2)-16|$$
$$|6-4|=|14-16|$$
$$|2|=|-2|$$
$$2=2$$

Solving inequalities with an absolute value

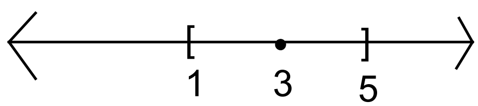
We will look at problems that can be summarized by the following:

$|x-a| < or \leq b$
$|x-a| > or \geq b$

Example:

$|5x+10| < 20$
$5|x+2| < 10$
$|x+2| < 2$
$|x-(-2)| < 2$

There are two ways to approach these problems.
First the visual way

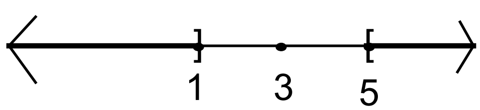


Notice that 3 is a distance of 2 from both 1 and 5. This interval can be described as follows:

$|x-3| \leq 2$

So this inequality is saying, any number a distance that is less than or equal to 2 from the number 3, which is the interval on the line. Notice that there is just one bounded interval. The solution can be written

$1 \leq x \leq 5$



$|x-3| \geq 2$

This inequality is saying, any number a distance greater than or equal to 2 from the number3. Notice from the graph that this is 2 unbounded intervals.

The solution to this inequality can be written

$x \leq 1 \; or \; x \geq 5$
This method will cover most of the problems you will see in class.

This alternative will also work and can be used in problems that the first method will not cover.

$|x-3| \le 2$

Consider the possibilities for what is inside the absolute value brackets.

Either

$$x-3 \ge 0$$
$$or$$
$$x-3 < 0$$

If $x-3 \ge 0$ we can rewrite the inequality as $x-3 \le 2$ which becomes

$$x \le 5$$

If on the other hand $x-3 < 0$ we have instead $-(x-3) \le 2$

Solving this we find

$$-(x-3) \le 2$$
$$x-3 \ge -2$$
$$x \ge 1$$

Combining these we get the same answer as before $1 \le x \le 5$

If instead of $|x-3| \le 2$ we have $|x-3| \ge 2$
we would get

$$x \le 1 \; or \; x \ge 5$$